Torus amplitudes and modular invariance

Otto T.P. Schmidt

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Abstract

In the study of string interactions, the ultimate objective will be the assignment of a probability to a certain process and the prediction of a physical cross section.
However, most string diagrams are not mere tree diagrams with a direct connection of incoming and outgoing strings. In order to obtain a more precise description of an interaction, one needs to consider intermediate processes such as self-energy corrections.
This report sets out to describe the simplest form of a self-energy correction, namely a one-loop closed string diagram. We therefore consider the simplest Riemann surface with non-zero genus, the torus, and show how it can be parameterised via the moduli space.
Following this analysis, the partition function for a single free compactified boson will be calculated and then generalised for the light-cone gauge with 24 transverse dimensions. As a result, we can obtain the amplitude for a one-loop interaction and show its modular invariance.

Contents

1 Introduction

The world-sheets of strings are two dimensional Riemann surfaces. Any interaction with a fixed number of in- and outgoing strings can be described in different levels of complexity. More complex interactions correspond to higher orders in the string perturbation expansion.
This work will focus on the simplest form of perturbation to a four closed string interaction, the one-loop interaction, for which the Riemann surface is a torus, i.e. a world-sheet with genus $g = 1$.

If a measure of the interaction probability is to be obtained, we need to find a way to parameterise different processes with the topology of a torus. This will be done by introducing the concept of a moduli space.
The analysis of the moduli space will produce a fundamental domain, which incorporates all possible parameterisations for a one-loop closed string interaction. More precisely, it contains all inequivalent forms of an interaction.
We will see that the moduli space is constructed by applying two conformal transformation or identifications to the upper half plane, which results in its segmentation into equivalent domains. This will also demonstrate that string theory is devoid of ultraviolet divergences.

The amplitude of a closed string interaction will be an integral over the moduli space, for which the integrand is the partition function. The partition function will count the number of states of the torus for a specific parameter of the moduli space.
Once both moduli space and partition function are determined, the amplitude can be postulated and its invariance under modular transformations can be shown.

2 The moduli space of tori

In the analysis of a one-loop closed string interaction we consider a toroidal Riemann surface. It is important to establish that the world-sheet is indeed a Riemann surface since the later notion of equivalent tori is derived from the equivalence of two Riemann surfaces via a conformal transformation.
After constructing the torus from a simple Riemann surface via conformal transformations, we will see that these conformal transformations restrict the set of inequivalent parameters.

2.1 One-loop open strings

Before approaching the moduli space of tori, consider a one-loop open string with light-cone momentum p^+ and world-sheet coordinates (τ, σ). This will serve as an intuitive analogon.

In light-cone gauge we have ($\beta = 2$ for open strings):

$$X^+ = 2\alpha' p^+ \tau \quad \text{and} \quad p^+ \sigma = \pi \int_0^\sigma d\tilde{\sigma} \mathcal{P}^{\tau +}(\tau, \tilde{\sigma}) \tag{1}$$

By choosing $\beta = \frac{1}{\alpha' p^+}$ we see that $\sigma \in [0, \frac{2\pi}{\beta}] = [0, 2\pi\alpha' p^+]$.

Therefore the light-cone diagram is:

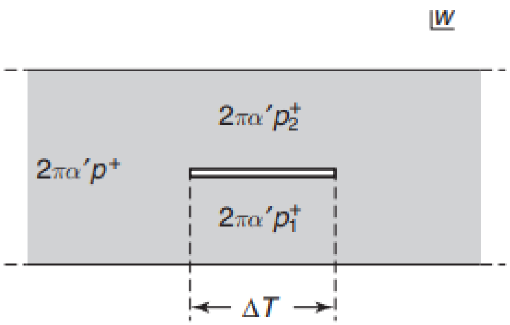


Figure 1: One-loop open string diagram with light-cone momentum p^+.[3]

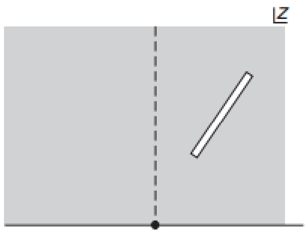
For fixed external momentum p^+ there are two parameters by which we can uniquely describe an interaction:

1. Interaction time $\Delta T \in (0, \infty)$
2. Intermediate string momentum $p_1^+ \in (0, p^+)$

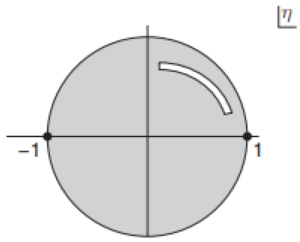
The class of Riemann surfaces of this process has two moduli. This however is not yet a statement about the moduli space.

To obtain the moduli space a canonical representation of the world-sheet diagram must be established. We can find a canonical representation by applying conformal transformations to the world-sheet diagram with coordinate $\omega = \tau + i\sigma$. These are shown in fig. 2a, 2b and 3.

The first transformation is an exponential map and the second a linear fractional transformation (LFT). The mapping from fig.2b to 3 is generally possible with a conformal transformation if the image is topologically an annulus.[3]
This process yields a canonical representation with a modulus r.



(a) Exponential map: $z = exp[\frac{w}{2\alpha' p^+}]$.[3]



(b) LFT: $\eta = \frac{1+iz}{1-iz}$.[3]

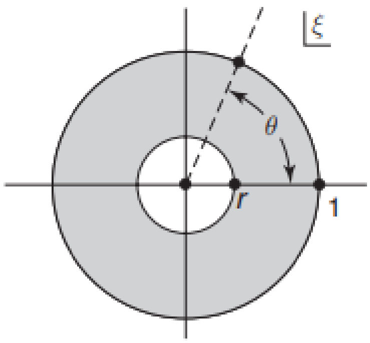


Figure 3: Canonical annulus with modulus r.[3]

2.2 Rectangular tori

As outlined before, we can only find the moduli space of a string interaction if the corresponding world-sheet diagram is a Riemann surface.
It is therefore essential for our analysis to show first that the torus, i.e. the world-sheet diagram of a one-loop closed string interaction, can be mapped conformally to a Riemann surface.

Consider a rectangular region of $\mathbb{C}$ as shown in fig. 4. We can now apply the analytic identifications $z \sim z + L_1$ and $z \sim z + iL_2$.
Here $z \sim z + L_1$ corresponds, intuitively speaking, to gluing the vertical edges together. Both ends of the resulting cylinder can then be identified via $z \sim z + iL_2$.
This is shown in fig. 5a and 5b.
These conformal transformations yield a torus. And since the torus is now linked via conformal transformation to a Riemann surface we can conclude that the torus is a Riemann surface as well.

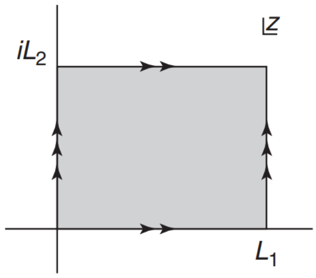
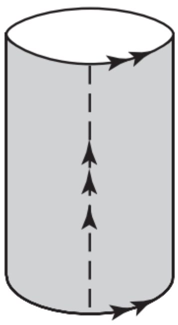
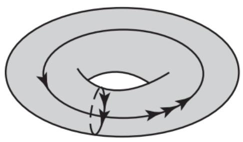


Figure 4: Region in $\mathbb{C}$.[3]



(a) First identification $z \sim z + L_1$.[3]



(b) Second identification $z \sim z + iL_2$.[3]

Although it seems that the region of the rectangular torus is defined by two parameters, i.e. the torus has two moduli, we can scale the identifications appropriately by choosing $z' = \frac{z}{L_1}$. We thereby obtain:

$$z' \sim z' + 1 \quad \text{and} \quad z' \sim z' + iT \quad \text{with} \quad T = \frac{L_2}{L_1} \tag{2}$$

This shows that the rectangular torus can be parameterised by one parameter alone.
However, so far this has only established the moduli itself and not the moduli space. Again, we can apply a series of conformal transformations. We choose a normalised complex coordinate $w = 1 + iT$:

1. $\tilde{w} = -iw$
2. $\eta = \frac{\tilde{w}}{T}$
3. $z = \eta + 2\frac{i}{T} = -\frac{iw}{T} + 2\frac{i}{T} = 1 + \frac{i}{T}$

Via these conformal transformations it becomes evident that the two Riemann surfaces with parameter T and $\frac{1}{T}$ are equivalent and hence tori with these parameters corresponds to equivalent one-loop interactions.

This has implications for the occurrence of ultraviolet divergences in string theory. For $T \to 0$ we consider short closed string interactions. For these cases the torus describes an infinitely fast self-interaction with infinite energies. This would then lead to ultraviolet divergences.
By the identification $T \sim \frac{1}{T}$ we can see that the problematic values of T, i.e. those leading to ultraviolet divergences, can also be interpreted differently, namely as long closed string interactions.
This leads to the conclusion that the range of values for the torus parameter T has to be split: if values $T \in [1, \infty)$ are included, the values $T \in (0, 1)$ should not be.
Here we choose the canonical interval $T \in [1, \infty)$ which only includes "long" tori.

2.3 General tori

Rectangular tori represent only a subset of all conformally inequivalent tori. A more general class of tori can be constructed.
For that we choose two complex numbers ω_1, ω_2 with $\mathrm{Im}(\frac{\omega_2}{\omega_1}) > 0$. A torus can again be obtained via the identifications $z \sim z + \omega_1$ and $z \sim z + \omega_2$.
Applying an appropriate scaling we obtain:

$$z \sim z + 1 \quad \text{and} \quad z \sim z + \tau \quad \text{with} \quad \tau = \frac{\omega_2}{\omega_1} \tag{3}$$

The newly introduced parameter τ is the modulus for general tori. The rectangular case can be obtained by setting $\mathrm{Re}(\tau) = 0$.
Note that until now the moduli space of a general torus is the upper half plane $\mathbb{H}$.

The region in $\mathbb{C}$ corresponding to a general torus is depicted in fig. 6 below.

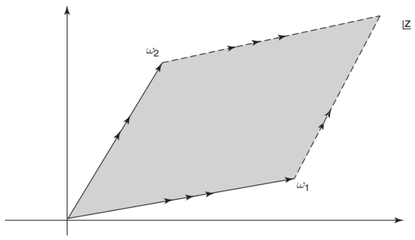
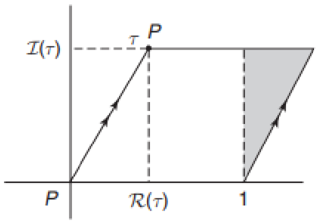
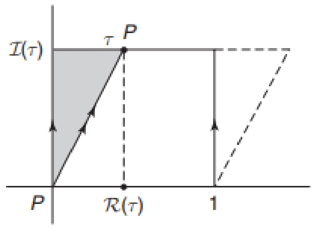


Figure 6: Region in $\mathbb{C}$ for a general torus.[3]

With a general torus we can consider the "twisting" of the torus. Intuitively, if a cylinder is twisted and the end surfaces are connected, we expect a different torus. In order to see what this intuitive notion of "twisting" implies for the moduli space of the torus, we need to apply conformal transformations which "twist" the torus. For this we take a canonical representation of a torus with unit length and $\mathrm{Re}(\tau) \neq 0$.
By applying $z \sim z + 1$, the shaded region in fig. 7a is identified with the shaded region if fig. 7b. We thereby obtained a rectangular region. The corresponding torus however is not rectangular. The difference can be seen if we now apply the identifications of the edges in order to obtain a cylinder. This is shown in fig. 8. It is clear that the points P (equivalent by $z \sim z + \tau$) are not orthogonal to each other on the cylinder.



(a) Applying $z \sim z + 1$.[3]



(b) Rectangular region.[3]

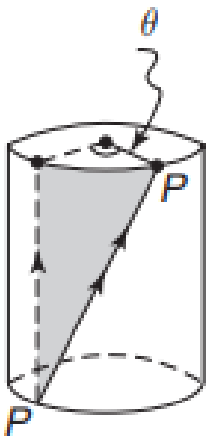


Figure 8: Cylinder with non-orthogonal points P.[3]

Following the illustrations we can introduce another parameter, the twisting angle θ. Since we considered a canonical representation of the general torus, the circumference of the cylinder is 1. It therefore follows:

$$\frac{\theta}{2\pi} = \frac{\text{Re}(\tau)}{1} \Longleftrightarrow \theta = 2\pi\,\text{Re}(\tau) \tag{4}$$

This formulation of the twisting parameter implies that $\tau \sim \tau + 1$ is an identification for the general torus. This follows since $\text{Re}(\tau+1) = \text{Re}(\tau)+1$ and $\theta \sim \theta+2\pi$.

The identification $\tau \sim \tau + 1$ is important since it restricts the parameter space of τ. We can now reduce the moduli space from $\mathbb{H}$ without loss of generality to

$$\mathcal{S}_0 = \{-\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2}, \text{Im}(\tau) > 0\} \tag{5}$$

2.4 Fundamental domain

The region $\mathcal{S}_0$ is a first candidate for the fundamental domain. The fundamental domain is defined as the set containing all inequivalent tori.
The study of the rectangular torus however hints that we have to restrict this canditate even more: for the rectangular torus we found that $T \sim \frac{1}{T}$. Since $\tau = iT$ we see that $\tau \sim -\frac{1}{\tau}$. Indeed this is another identification leading to equivalent tori.
There are two modular transformations so far:

1. T-modular transform: $\tau \sim \tau + 1$
2. S-modular transform: $\tau \sim -\frac{1}{\tau}$

We can restrict ourselfs to those two transformation since the later defined modular group is spanned by those two transformation. All other modular transformations, i.e. group actions of the modular group, can be obtained by a suitable combination of T- and S-modular transforms.

The S-modular transform identifies points in $|\tau| < 1$ with points in $|\tau| > 1$. This restricts the candidate $\mathcal{S}_0$ to

$$\mathcal{F}_0 = \{-\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2}, \text{Im}(\tau) > 0, |\tau| \geq 1 \text{ and } \text{Re}(\tau) \geq 0 \text{ if } |\tau| = 1\} \tag{6}$$

The candidate $\mathcal{F}_0$ for the fundamental domain contains three parts:

1. $[-\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2}, \text{Im}(\tau) > 0]$ is the domain $\mathcal{S}_0$
2. $|\tau| \geq 1$ incorporates that points in $|\tau| < 1$ are identified with points in $|\tau| > 1$
3. $\text{Re}(\tau) \geq 0$ if $|\tau| = 1$

Point three needs a separate analysis: consider $\tau = a + ib$ with $|\tau| = 1$ in $\mathcal{S}_0$. Assume first that $a \in (0, \frac{1}{2})$. Then:

$$-\frac{1}{\tau} = -\frac{1}{a + ib} = -a + ib \tag{7}$$

We can conclude that τ on the unit circle in $\mathcal{S}_0$ with $a \in (0, \frac{1}{2})$ can be identified to a torus with $a \in (-\frac{1}{2}, 0)$. Hence $\mathcal{F}_0$ needs to include

$$\text{Re}(\tau) \geq 0 \text{ if } |\tau| = 1 \tag{8}$$

if we want to exclude equivalent tori in the domain.

In the construction of $\mathcal{F}_0$ we made an arbitrary choice, namely to consider only $|\tau| \geq 1$. Although this choice was made without loss of generality, there is a subtle motivation for it.
To see that take the torus $\tau = i$. Via the T-modular transform we see that there is an infinite set of equivalent tori τ_n outside of $\mathcal{S}_0$:

$$\tau_n = i + n \text{ for } n \geq 1 \tag{9}$$

Applying the S-modular transform yields:

$$-\frac{1}{\tau_n} = -\frac{n}{n^2+1} + \frac{i}{n^2+1} \tag{10}$$

These tori have $\mathrm{Re}(-\frac{1}{\tau_n}) \in [-\frac{1}{2}, 0)$ and $|-\frac{1}{\tau_n}| < 1$. Therefore $\tau_n \in \mathcal{S}_0 - \mathcal{F}_0$. We can conclude that $\mathcal{S}_0 - \mathcal{F}_0$ contains infinitely many copies of $\tau = i$. It is therefore natural to exclude this region. Hence the choice for $|\tau| \geq 1$.

The domain $\mathcal{F}_0$ is illustrated in fig. 9

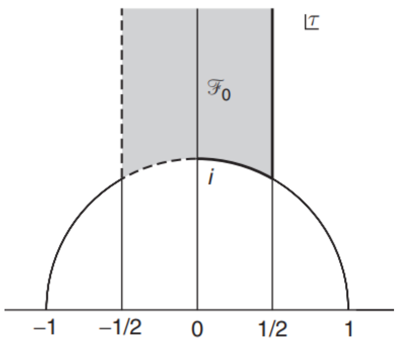


Figure 9: Fundamental domain $\mathcal{F}_0$.[3]

It is important to stress that this fundamental domain is merely a choice for the set of inequivalent tori. In fact, there are infinitely many fundamental domains, i.e. infinitely many representations of the moduli space. Since T- and S-modular transforms do not change a torus τ but only give an equivalent representation, the infinite set of fundamental domains arises by applying combinations of T- and S-modular transforms to $\mathcal{F}_0$. This can be seen in fig. 10.

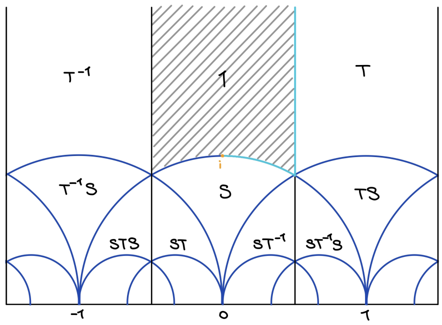


Figure 10: Fundamental domains for T- and S-modular transforms.

As for now, the notion modular has so far been used without justification. It is derived from the modular group $PSL(2,\mathbb{Z})$, which will be derived below.
Consider a general fractional linear transformation $g \in G$, with G a set of transformations, such that:

$$g\tau = \frac{a\tau + b}{c\tau + d}, \ \mathrm{Im}(g\tau) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} \tag{11}$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.
Equivalently we can use a matrix representation:

$$[g] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det[g] = 1 \tag{12}$$

The group elements $g \in G$ satisfy the group homomorphism $\phi : G \to G$, $[g_1 g_2] \mapsto [g_1][g_2]$. This can be easily verified using the matrix representations.
The set of transformations G is then called the modular group $PSL(2,\mathbb{Z})$.

We can now reinterpret the known T- and S-modular transformations via the matrix notation:

$$[T] = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad [S] = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \tag{13}$$

In order to substantiate the claim that $\mathcal{F}_0$ is the fundemental domain, we can show that for all $\tau \in \mathbb{H}$ there exists $g \in G'$ such that $g\tau \in \mathcal{F}_0$. Here G' is the set of all combinations of T- and S-modular transforms.
We can go about this in three steps:

1. Show that for each $\tau \in \mathbb{H}$ there exists a $g \in G'$ such that $\text{Im}(g\tau)$ is largest.
2. Show that $\tau' = T^n g\tau \in \mathcal{S}_0$ really is in $\bar{\mathcal{F}}_0$.
3. Show that $\tau \in \bar{\mathcal{F}}_0$ can be send to $\tau \in \mathcal{F}_0$ via T- or S-transforms.

1.: It is equivalent to show that $|c\tau + d|$ is smallest since $\text{Im}(g\tau) = \frac{\text{Im}(\tau)}{|c\tau+d|^2}$ for a transformation $g \in G'$. For that we notice that $(c, d) \in \mathbb{Z}^2$ span a lattice. So for a fixed $\tau \in \mathbb{H}$ there exists only a finite number of points in the lattice such that for $\alpha > 0$ we have $|c\tau + d| < \alpha$. However, the transformation g is not unique.

2.: Consider a torus $\tau' = T^n g\tau$ with n such that $\tau' \in \mathcal{S}_0$. We know that $\text{Im}(\tau') = \text{Im}(g\tau)$ since $c = 0$ and $d = 1$ for the T-modular transform.
For any $g' \in G'$ we also know from 1. that $\text{Im}(g'\tau') \leq \text{Im}(\tau')$.
Now assume that $\tau' \notin \bar{\mathcal{F}}_0$. This implies $|\tau'| < 1$. Suppose $g' = S$. We then have:

$$\text{Im}(S\tau') = \frac{\text{Im}(\tau')}{|\tau'|^2} > \text{Im}(\tau') \tag{14}$$

This is a contradiction since $\text{Im}(\tau')$ should be largest. Hence we can conclude $\tau \in \bar{\mathcal{F}}_0$.

3.: If $\tau \in \bar{\mathcal{F}}_0$ is already in $\mathcal{F}_0$ nothing remains to be shown. If τ however lies on the boundary of $\mathcal{F}_0$ we can simply apply a T- or S-modular transform once to send τ to $\mathcal{F}_0$.

3 Torus partition function

This section will establish the partition function of the torus. The partition function will serve as the integrand of the later postulated torus amplitude.
It will be shown how the partition function behaves under modular transforms and that it is indeed modular invariant.

3.1 Single free boson

First we consider the case of a single free boson with a compactified coordinate $X \sim X + 2\pi r$. Hence the boundary condidtions read:

$$X(z+\tau, \bar{z}+\bar{\tau}) = X(z,\bar{z}) + 2\pi r n'$$
$$X(z+1, \bar{z}+1) = X(z,\bar{z}) + 2\pi r n$$

with $n, n' \in \mathbb{Z}$.
The solution to the classical equation of motion $\partial\bar{\partial}X = 0$ is:

$$X^{n,n'}(z,\bar{z}) = 2\pi r \frac{1}{2i\tau_2}[n'(z-\bar{z}) + n(\tau\bar{z} - \bar{\tau}z)] \tag{15}$$

The action is defined as

$$S = \frac{1}{2\pi}\int \partial X \bar{\partial} X \tag{16}$$

In a path integral formalism, the partition function or propagator kernel is then obtained as [2]:

$$\int e^{-S} = 2\pi r \frac{\sqrt{2\tau_2}}{\pi} \frac{1}{\det'^{\frac{1}{2}} \Box} \sum_{n,n' \in \mathbb{Z}} e^{-S[X^{n,n'}]}$$
$$= \frac{1}{\eta\bar{\eta}} \sum_{n,m \in \mathbb{Z}} e^{2\pi i \tau \frac{1}{2}(\frac{p}{2}+w)^2} e^{-2\pi i \bar{\tau} \frac{1}{2}(\frac{p}{2}-w)^2}$$

where $\tau = \tau_1 + i\tau_2$, $p = \frac{m}{r}$ and $w = nr$.

The Dedekind eta function $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ with $q = e^{2\pi i\tau}$ appears since:

$$\begin{aligned}
\det{}'\Box &= \prod_{\{m,n\}\neq\{0,0\}} \frac{\pi^2}{\tau_2^2}(n-m\tau)(n-m\bar{\tau}) \\
&= \frac{\pi^2}{\tau_2^2}(2\pi)^2 \prod_{m>0,n\in\mathbb{Z}}(n-m\tau)(n+m\tau)(n+\bar{\tau})(n-\bar{\tau}) \\
&= 4\tau_2^2(q\bar{q})^{\frac{1}{12}} \prod_{m>0}(1-q^m)^2(1-\bar{q}^m)^2 \\
&= 4\tau_2^2\eta^2\bar{\eta}^2
\end{aligned}$$

with $\Box = -\partial\bar{\partial}$ and $\det'$ the regularised determinant (omitting the case $m = n = 0$).

So the partition function for the free boson compactified on a circle with radius r reads:

$$Z_r(\tau,\bar{\tau}) = \int e^{-S} = \frac{1}{|\eta|^2}\sum_{m,n} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} \tag{17}$$

where

$$p_L = \frac{m}{2r} + nr \text{ and } p_R = \frac{m}{2r} - nr \tag{18}$$

In order to understand this result we need to consider the Hilbert space of the single free boson on a compactified world-sheet as a product Hilbert space:

$$\mathcal{H} = \mathcal{H}_{osc.} \otimes \bigoplus_{p,w} \mathcal{H}_{p,w} \tag{19}$$

where $\mathcal{H}_{osc.}$ is the bosonic Fock space generated by the mode operators α_{-n} and $\bigoplus_{p,w}\mathcal{H}_{p,w}$ is the Hilbert space for different winding and momentum states, i.e. the compactification states.
We can expect that the partition function splits into an oscillation part and a part for the compactification numbers p, w in a similar way:

$$Z = Z_{osc.} \times \sum_{p,w} Z_{p,w} \tag{20}$$

It is therefore natural to interpret the partition function in eq. 17 in the following way:

$$Z_{osc.} = \frac{1}{|\eta|^2}$$
$$Z_{p,w} = q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}$$

3.2 Modular invariance of the partition function

The modular invariance of the partition function is the essential property for the modular invariance of the amplitude.
To fully understand the behaviour of the partition function under T- or S-modular transformations, we need to understand the behaviour of its components.

The Dedekind eta function is a modular form with weight $\frac{1}{2}$ and transforms under T- or S-modular transformations like

$$\begin{aligned}\eta(\tau+1) &= e^{2\pi i(\tau+1)\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n e^{2\pi i n}) \\ &= e^{i\frac{\pi}{12}}\eta(\tau) \\ \eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\eta(\tau)\end{aligned}$$

Let us first consider the whole partition function under T-modular transformations:

$$\begin{aligned}Z_r(\tau+1,\bar{\tau}+1) &= \frac{1}{|e^{i\frac{\pi}{12}}\eta|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2} e^{2\pi i\frac{1}{2}(\frac{m}{2r}+nr)^2}\bar{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2}e^{-2\pi i\frac{1}{2}(\frac{m}{2r}-nr)^2} \\ &= \frac{1}{|\eta|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{2r}+nr)^2}\bar{q}^{\frac{1}{2}(\frac{m}{2r}-nr)^2}e^{2\pi i mn} \\ &= Z_r(\tau,\bar{\tau})\end{aligned}$$

We can conclude T-modular invariance.

The behavior under S-modular transform is more difficult. Its is necessary to first introduce a mathematical method, the Poisson resummation.
Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function $f \in \mathcal{S}(\mathbb{R})$. Then:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n) \tag{21}$$

Consider functions of the form $e^{-\pi a q^2}$. Then:

$$\sum_{q\in\Sigma} e^{-\pi a q^2} = \frac{1}{a^{1/2}} \sum_{p\in\Sigma^*} e^{-\frac{\pi}{a}p^2} \tag{22}$$

where Σ is a lattice over $\mathbb{Z}$. We can therefore assume that it is self-dual, i.e. $\Sigma^* = \Sigma$.
We then have for the partition function:

$$\begin{aligned}
Z_r(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}) &= \frac{1}{|\tau||\eta|^2} \sum_{p_L\in\Sigma} e^{-\pi(\frac{1}{-i\tau})p_L^2} \sum_{p_R\in\bar{\Sigma}} e^{-\pi(\frac{1}{i\bar{\tau}})p_R^2} \\
&= \frac{1}{|\tau||\eta|^2} \sum_{p_L\in\Sigma^*} \sqrt{-i\tau} e^{i\pi\tau p_L^2} \sum_{p_R\in\bar{\Sigma}^*} \sqrt{i\bar{\tau}} e^{-i\pi\bar{\tau}p_R^2} \\
&= \frac{1}{|\tau||\eta|^2} |\tau| \sum_{p_L\in\Sigma, p_R\in\bar{\Sigma}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \\
&= Z_r(\tau, \bar{\tau})
\end{aligned}$$

This concludes S-modular invariance for the partiton function.

Before approaching a different derivation for the partition function, let us consider the path integral result from before and observe the behaviour for $r \to \infty$. This can be seen as the decompactification of the compactified bosonic string.
If we do so, the momenta p_L, p_R become continuous $p_L = p_R = \frac{m}{2r}$. The contribution from winding $w = nr$ leads to a fast oscillating factor in the integral and hence vanishes.

We have:

$$
\begin{aligned}
Z_r(\tau,\bar{\tau}) &= \frac{1}{|\eta|^2}\int_{-\infty}^{\infty} dk q^{\frac{1}{2}k^2}\bar{q}^{\frac{1}{2}k^2} \\
&= \frac{1}{|\eta|^2}\int_{-\infty}^{\infty} dk e^{2\pi i\frac{1}{2}k^2\tau}e^{-2\pi i\frac{1}{2}k^2\bar{\tau}} \\
&= \frac{1}{|\eta|^2}\int_{-\infty}^{\infty} dk e^{2\pi i\frac{1}{2}k^2 2i\,\mathrm{Im}\{\tau\}} \\
&= \frac{1}{|\eta|^2}\int_{-\infty}^{\infty} dk e^{-2\pi k^2\,\mathrm{Im}\{\tau\}} \\
&\propto \frac{1}{|\eta|^2}\frac{1}{\mathrm{Im}(\tau)^{\frac{1}{2}}}
\end{aligned}
$$

With the interpretation of eq. 20 we can see that

$$
\begin{aligned}
Z_{osc.} &= \frac{1}{|\eta|^2} \\
Z_{p,w} &= \frac{1}{\mathrm{Im}(\tau)^{\frac{1}{2}}}
\end{aligned}
$$

3.3 Partition function via state trace

An alternative approach to obtain the partition function is the state trace.

The compactification states are counted via the results we obtained from the path integral. No trace is involved.

$$
\sum_{p_L\in\Sigma,p_R\in\bar{\Sigma}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2} \xrightarrow{r\to\infty} \int_{-\infty}^{\infty} dk q^{\frac{1}{2}k^2}\bar{q}^{\frac{1}{2}k^2} \propto \frac{1}{\mathrm{Im}(\tau)^{\frac{1}{2}}} \tag{23}
$$

A possible way of counting the oscillation states however does involve the state trace and the zero mode Virasoro operators $L_0, \bar{L}_0$:

$$
\sum_{\Phi} \langle\Phi|\, e^{2\pi i\tau L_0}e^{-2\pi i\bar{\tau}\bar{L}_0}\,|\Phi\rangle = \mathrm{Tr}\big(q^{L_0}\bar{q}^{L_0}\big) \tag{24}
$$

The form of the propagator, i.e. $q^{L_0}\bar{q}^{L_0}$, is motivated by the two following observations:

1.

$$\begin{aligned}\mathrm{Tr}\left(q^{L_0}\right) &\propto \mathrm{Tr}\left(q^{\sum_{k=1}^{\infty} k\hat{N}}\right)\\ &= \prod_{k=1}^{\infty} \mathrm{Tr}\left(q^{k\hat{N}}\right)\\ &= \prod_{k=1}^{\infty}\sum_{n\geq 0} \langle n|\, q^{k\hat{N}}\, |n\rangle\\ &= \prod_{k=1}^{\infty}\sum_{n\geq 0} (q^k)^n\\ &= \prod_{k=1}^{\infty} \frac{1}{1-q^k}\end{aligned}$$

This is the partition function of a boson from the Bose-Einstein distribution. With a correction factor we get the eta function:

$$\mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\right) = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{\eta(\tau)} \tag{25}$$

2.

$$\begin{aligned}q^{L_0}\bar{q}^{\bar{L_0}} &= e^{2\pi i(\tau_1+i\tau_2)L_0} e^{-2\pi i(\tau_1-i\tau_2)\bar{L_0}}\\ &= e^{2\pi i\tau_1(L_0-\bar{L_0})} e^{-2\pi\tau_2(L_0+\bar{L_0})}\end{aligned}$$

Here $L_0 - \bar{L}_0$ is the momentum P which generates translation in σ. Similarly $L_0 + \bar{L}_0$ is the Hamiltonian H which generates τ translation.
The propagator has then the interpretation of a τ and σ sweep.

Equation 24 therefore evaluates as

$$\mathrm{Tr}\left(q^{L_0}\bar{q}^{L_0}\right) = \frac{1}{|\eta(\tau)|^2} \tag{26}$$

and together with eq. 23 the partition function reads (as in the the case for $r \to \infty$):

$$Z(\tau,\bar{\tau})_{1d} = \frac{1}{|\eta(\tau)|^2}\frac{1}{(\mathrm{Im}\,\tau)^{\frac{1}{2}}} \tag{27}$$

Generalised for the 24 transverse dimensions in light-cone gauge we then obtain:

$$Z(\tau,\bar{\tau})_{l.c.} = (\frac{1}{|\eta(\tau)|^2}\frac{1}{(\mathrm{Im}\,\tau)^{\frac{1}{2}}})^{24} = \frac{1}{|\eta(\tau)|^{48}}\frac{1}{(\mathrm{Im}\,\tau)^{12}} \tag{28}$$

This is the form we will use in the amplitude.

4 Modular invariance of the torus amplitude

The correct one-loop vacuum amplitude reads [1]:

$$A_0^{g=1} \propto \int_{\mathcal{F}_0} \frac{d^2\tau}{4(\mathrm{Im}(\tau))^2} Z(\tau, \bar{\tau}) \tag{29}$$

This amplitude carries the correct physical intuition: each possible form of a one-loop interaction is parametrised by τ. To get a probability measure for a process we therefore need to weight every $\tau \in \mathcal{F}_0$ with the partition function $Z(\tau, \bar{\tau})$.

We can rewrite the amplitude using the light-cone Hamiltonian and momentum (with $\tau = \tau_1 + i\tau_2$) [1]:

$$A_0^{g=1} \propto \int_{\mathcal{F}_0} \frac{d^2\tau}{16\pi^2\alpha'\tau_2^2} \int \frac{d^{24}p}{(2\pi)^{24}} \operatorname{Tr}\left(e^{-2\pi\tau_2 H_{l.c.}} e^{-2\pi i\tau_1 P_{l.c.}}\right) \tag{30}$$

This form yields an interpretation of the amplitude in terms of the light-cone Hamiltonian and momentum.
Consider $\tau_1 = 0$. In this case τ_2 plays the role of a Euclidean time. We know that $\tau_1 = 0$ corresponds to a rectangular torus.
The partition function counts the number of states propagating around the torus in τ_2 direction and weights them with $e^{-2\pi\tau_2 H_{l.c.}}$.

Now consider a cylinder with length τ_2 and $\tau_1 \neq 0$ whose ends are identified. We can twist the ends by the twist angle $\theta = 2\pi\tau_1$. This twist is then induced by $P_{l.c.}$.

To show that the form in eq. 29 is modular invariant, we need to establish modular invariance for the partition function Z, the integration domain $\mathcal{F}_0$ and the measure $\frac{d^2\tau}{4(\mathrm{Im}(\tau))^2}$.
The modular invariance of the partition function has been shown in the previous section. Furthermore we know that the integration domain is by definition modular invariant, the fundamental domain does not change under T- or S-modular transformations.

The remaining thing is to show modular invariance of the measure:

$$\begin{aligned}
d(g\tau) &= [\frac{a(c\tau+d)-(a\tau+b)c}{(c\tau+d)^2}]d\tau \\
&= [\frac{ad-bc}{(c\tau+d)^2}]d\tau \\
&= |c\tau+d|^{-2}d\tau \\
\operatorname{Im}(g\tau) &= |c\tau+d|^{-2}\operatorname{Im}(\tau)
\end{aligned}$$

Therefore the measure transforms like:

$$\frac{d^2(g\tau)}{\operatorname{Im}(g\tau)^2} = \frac{|c\tau+d|^{-4}d^2\tau}{(|c\tau+d|^{-2}\operatorname{Im}(\tau))^2} = \frac{d^2\tau}{\operatorname{Im}(\tau)^2} \tag{31}$$

This shows that the measure is also modular invariant and we can conclude the modular invariance of the amplitude for a one-loop closed string interaction.

5 Conclusion

This report set out to derive a form for the amplitude of the one-loop closed string interaction. It was established how the world-sheet diagram of such a process, the torus, defines the interaction and how it is connected to a simple Riemann surface. The analysis of the parameter space produced a set of parameters τ which correspond to inequivalent interactions, i.e. different tori. We have called this set the fundamental domain or the moduli space of the torus. To define the moduli space properly we have introduced the modular group $PSL(2, \mathbb{Z})$.

In a next step the partition function was postulated and motivated. This function is most important to characterise the states over which the amplitude will integrate. It therefore also needs to be modular invariant.
Lastly, the amplitude was defined and its modular invariance was shown. This leaves us with a tool to calculate the probabilites of different one-loop interactions.

However, to fully understand scattering or interaction events, more orders of perturbation need to be included. Here we have focused only on the first order. This study will then include world-sheets with multiple geni and higher dimensional moduli spaces.

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